

# How can we check the uncertainty relation?

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**Abstract.** The state-extended uncertainty relations generalizing the Robertson Schrödinger inequality are presented in the form appropriate for the experimental check by homodyne photon state detection. The method of qubit portrait of qudit states identified with the tomographic probability distributions is discussed to analyze the entanglement of two-mode field.

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## 1. Introduction

There are recent results where the authors discuss the quantum theory going beyond conventional quantum mechanics [1–3]. In this connection the precise experimental check of basic quantum phenomena with high accuracy, e.g. quantum uncertainty relations like Heisenberg position–momentum uncertainty relation [4], Robertson [5] and Schrödinger [6] uncertainty relations, purity dependent uncertainty relations [7, 8] and other quantum inequalities would be interesting to fulfill. The new formulation of quantum mechanics based on tomographic probability representation of quantum states [9–11] provides convenient tools to suggest such experiments [12] using the homodyne photon state detection where optical tomograms of the photon quantum states are measured [13]. In [14, 25] new quantum uncertainty relations were found. In contrast to Heisenberg and Robertson-Schrödinger uncertainty relations Trifonov inequalities depend on two quantum states and they were called state-extended uncertainty relations. In fact these inequalities provide a generalization of standard position-momentum uncertainty relations. The state-extended generalization of Heisenberg uncertainty relations was studied in [16] and its tomographic form was found and proposed for experimental check in the photon homodyne detection. In this work we consider another state-extended generalization of position-momentum uncertainty relations. Our aim is to obtain tomographic form of Trifonov inequality which is state-extended Robertson-Schrödinger uncertainty relations containing covariance of position and momentum. Another our goal is to review probability representation approach in the context of studying qubit portrait of qudit states [17, 18]. The paper is organized as follows. In next section 2 we present the optical tomography scheme of one-mode quantum electromagnetic field. (see, e.g. [9]). In Sec.3 we give Trifonov inequalities in

tomographic form. In Sec.4 we discuss qubit portrait method and in Sec.5 present conclusions and prospects.

## 2. Optical tomography

The quantum state in probability representation of quantum mechanics is determined by optical tomogram  $w(X, \theta)$ . Here  $-\infty < X < +\infty$ ,  $0 \leq \theta \leq 2\pi$ . The optical tomogram is probability density of random homodyne quadrature  $X$ . It depends on the angle  $\theta$  which in quantum optics is called local oscillator phase. The optical tomogram provides the density operator  $\hat{\rho}$  of the photon quantum state

$$\hat{\rho} = \frac{1}{2\pi} \int_0^\pi d\theta \int_{-\infty}^{+\infty} d\eta dX \omega(X, \theta) |\eta| \exp i\eta(X - \hat{q} \cos \theta - \hat{p} \sin \theta). \quad (1)$$

The optical tomogram can be found if the density operator  $\hat{\rho}$  is known

$$w(X, \theta) = \text{Tr} \hat{\rho} \delta(X - \cos \theta \hat{q} - \sin \theta \hat{p}). \quad (2)$$

The physical meaning of the optical tomogram is the following one. It is nonnegative probability density of the homodyne quadrature

$$X = \hat{q} \cos \theta + \hat{p} \sin \theta. \quad (3)$$

Consequently for  $\theta = 0$  the tomogram in quantum optics provides the probability distribution of first quadrature  $q$  and for  $\theta = \pi/2$  the tomogram yields the probability distribution of the second quadrature  $p$ . In quantum mechanics for  $\theta = 0$  and  $\theta = \pi/2$  the tomogram provides probability distributions of position and momentum, respectively. The most important property of the optical tomogram is that it is measured experimentally [19–21]. For pure state with wave function  $\Psi(y)$  the tomogram reads

$$w(X, \theta) = \frac{1}{2\pi |\sin \theta|} \left| \int \Psi(y) \exp \left( \frac{iy^2}{2 \tan \theta} - \frac{iXy}{\sin \theta} \right) dy \right|^2. \quad (4)$$

If the Hamiltonian  $\hat{H} = \hat{p}^2/2 + U(\hat{q})$ , the optical tomogram obeys the evolution equation of the form [22]

$$\begin{aligned} \frac{\partial}{\partial t} w(X, \theta, t) = & \left[ \cos^2 \theta \frac{\partial}{\partial \theta} - \frac{1}{2} \sin 2\theta \left( 1 + X \frac{\partial}{\partial X} \right) \right] w(X, \theta, t) \\ & + \frac{1}{i} \left\{ V \left[ \left( \sin \theta \frac{\partial}{\partial \theta} \left( \frac{\partial}{\partial X} \right)^{-1} + X \cos \theta + i \frac{\sin \theta}{2} \frac{\partial}{\partial X} \right) \right] \right. \\ & \left. - V \left[ \left( \sin \theta \frac{\partial}{\partial \theta} \left( \frac{\partial}{\partial X} \right)^{-1} + X \cos \theta - i \frac{\sin \theta}{2} \frac{\partial}{\partial X} \right) \right] \right\} w(X, \theta, t). \end{aligned} \quad (5)$$

## 3. Uncertainty relations

In view of physical meaning of the optical tomogram the Heisenberg uncertainty relation

$$\sigma_{qq} \sigma_{pp} \geq 1/4 \quad (6)$$

can be presented in the tomographic form as [23]

$$\begin{aligned} & \left[ \int w(X, 0) X^2 dX - \left( \int w(X, 0) X dX \right)^2 \right] \\ & \times \left[ \int w(X, \pi/2) X^2 dX - \left( \int w(X, \pi/2) X dX \right)^2 \right] \geq \frac{1}{4}. \end{aligned} \quad (7)$$

The Robertson-Schrödinger inequality

$$\sigma_{qq}\sigma_{pp} - \sigma_{qp}^2 \geq 1/4. \quad (8)$$

where quadrature variances and covariance are calculated for the same state was generalized by Trifonov [24, 25]. For two pure states  $|\Psi_1\rangle, |\Psi_2\rangle$  this state-extended inequality reads

$$\begin{aligned} & \frac{1}{2} [\text{Tr}(\hat{q}^2 |\Psi_1\rangle\langle\Psi_1|) - (\text{Tr}(\hat{q} |\Psi_1\rangle\langle\Psi_1|))^2] \\ & \times [\text{Tr}(\hat{p}^2 |\Psi_2\rangle\langle\Psi_2|) - (\text{Tr}(\hat{p} |\Psi_2\rangle\langle\Psi_2|))^2] \\ & + \frac{1}{2} [\text{Tr}(\hat{q}^2 |\Psi_2\rangle\langle\Psi_2|) - (\text{Tr}(\hat{q} |\Psi_2\rangle\langle\Psi_2|))^2] \\ & \times [\text{Tr}(\hat{p}^2 |\Psi_1\rangle\langle\Psi_1|) - (\text{Tr}(\hat{p} |\Psi_1\rangle\langle\Psi_1|))^2] \\ & - \left\{ \text{Tr} \left( \frac{\hat{q}\hat{p} + \hat{p}\hat{q}}{2} |\Psi_2\rangle\langle\Psi_2| \right) - \text{Tr}(\hat{q} |\Psi_2\rangle\langle\Psi_2|) \text{Tr}(\hat{p} |\Psi_2\rangle\langle\Psi_2|) \right\} \\ & \times \left\{ \text{Tr} \left( \frac{\hat{q}\hat{p} + \hat{p}\hat{q}}{2} |\Psi_1\rangle\langle\Psi_1| \right) - \text{Tr}(\hat{q} |\Psi_1\rangle\langle\Psi_1|) \text{Tr}(\hat{p} |\Psi_1\rangle\langle\Psi_1|) \right\} \geq \frac{1}{4}. \end{aligned} \quad (9)$$

This inequality can be written in the tomographic form and it reads

$$\begin{aligned} & \frac{1}{2} \left[ \int w_1(X, \theta) X^2 dX - \left( \int w_1(X, \theta) X dX \right)^2 \right] \\ & \times \left[ \int w_2(X, \theta + \pi/2) X^2 dX - \left( \int w_2(X, \theta + \pi/2) X dX \right)^2 \right] \\ & + \frac{1}{2} \left[ \int w_2(X, \theta) X^2 dX - \left( \int w_2(X, \theta) X dX \right)^2 \right] \\ & \times \left[ \int w_1(X, \theta + \pi/2) X^2 dX - \left( \int w_1(X, \theta + \pi/2) X dX \right)^2 \right] \\ & - \left\{ \int w_1(X, \theta + \frac{\pi}{4}) X^2 dX - \left( \int w_1(X, \theta + \frac{\pi}{4}) X dX \right)^2 \right. \\ & - \frac{1}{2} \left[ \int w_1(X, \theta) X^2 dX - \left( \int w_1(X, \theta) X dX \right)^2 \right] \\ & - \frac{1}{2} \left[ \int w_1(X, \theta + \frac{\pi}{2}) X^2 dX - \left( \int w_1(X, \theta + \frac{\pi}{2}) X dX \right)^2 \right] \Big\} \\ & \times \left\{ \int w_2(X, \theta + \frac{\pi}{4}) X^2 dX - \left( \int w_2(X, \theta + \frac{\pi}{4}) X dX \right)^2 \right. \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2} \left[ \int w_2(X, \theta) X^2 dX - \left( \int w_2(X, \theta) X dX \right)^2 \right] \\
& -\frac{1}{2} \left[ \int w_2(X, \theta + \frac{\pi}{2}) X^2 dX - \left( \int w_2(X, \theta + \frac{\pi}{2}) X dX \right)^2 \right] \} \geq 1/4. \quad (10)
\end{aligned}$$

The obtained inequalities can be checked if both tomograms  $w_1(X, \theta)$  and  $w_2(X, \theta)$  are measured. This inequality takes place for mixed state too.

#### 4. Qubit portrait and inequalities for optical tomograms

Qubit portrait of qudit states provides the probability distribution given by two positive numbers  $p_1, p_2$ , where  $p_1 + p_2 = 1$  obtained from an initial probability distribution  $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_N$  where  $\sum_k \mathcal{P}_k = 1$ . This qubit probability distribution can be obtained using linear map of  $N$ -vector with components  $\mathcal{P}_k$  onto two-vector with components  $p_1, p_2$ . The map can be described e.g. by the corresponding stochastic matrix. If one has probability density  $w(X, \theta)$  the qubit portrait can be also constructed by using the rectangular matrix

$$p_m(\theta) = \int K_m(X) w(X, \theta) dX, \quad m = \frac{1}{2}, -\frac{1}{2} \quad (11)$$

where  $w(X, \theta)$  is the tomogram of a quantum state. If one has two-mode state with the optical tomogram  $w(X_1, X_2, \theta_1, \theta_2)$  the generalized qubit portrait provides the analog of spin-tomogram of two-qubits

$$p(m_1, m_2, \theta_1, \theta_2) = \int K_{m_1 m_2}(X_1, X_2) w(X_1, X_2, \theta_1, \theta_2) dX_1 dX_2. \quad (12)$$

For example the matrix  $K_{m_1 m_2}(X_1, X_2)$  can have factorized form. The quantum correlations for the two-mode states can be studied by considering the properties of the function (12). For example the probability four-vector  $\vec{p}(\theta_1, \theta_2)$  depending on extra angle parameters can be studied analogously to the case of studying spin tomographic probability of two-qubit entangled state for which the Bell inequality violation is sufficient condition of the state entanglement. Then the Bell number is given in terms of the function  $p(m_1, m_2, \theta_1, \theta_2)$  as follow

$$\begin{aligned}
B = \max & |p_{+\frac{1}{2}+\frac{1}{2}}(\theta_1, \theta_2) - p_{+\frac{1}{2}-\frac{1}{2}}(\theta_1, \theta_2) - p_{-\frac{1}{2}+\frac{1}{2}}(\theta_1, \theta_2) + p_{-\frac{1}{2}-\frac{1}{2}}(\theta_1, \theta_2) \\
& + p_{+\frac{1}{2}+\frac{1}{2}}(\theta_1, \theta_3) - p_{+\frac{1}{2}-\frac{1}{2}}(\theta_1, \theta_3) - p_{-\frac{1}{2}+\frac{1}{2}}(\theta_1, \theta_3) + p_{-\frac{1}{2}-\frac{1}{2}}(\theta_1, \theta_3) \\
& + p_{+\frac{1}{2}+\frac{1}{2}}(\theta_4, \theta_2) - p_{+\frac{1}{2}-\frac{1}{2}}(\theta_4, \theta_2) - p_{-\frac{1}{2}+\frac{1}{2}}(\theta_4, \theta_2) + p_{-\frac{1}{2}-\frac{1}{2}}(\theta_4, \theta_2) \\
& - p_{+\frac{1}{2}+\frac{1}{2}}(\theta_4, \theta_3) + p_{+\frac{1}{2}-\frac{1}{2}}(\theta_4, \theta_3) + p_{-\frac{1}{2}+\frac{1}{2}}(\theta_4, \theta_3) - p_{-\frac{1}{2}-\frac{1}{2}}(\theta_4, \theta_3)|. \quad (13)
\end{aligned}$$

For factorized matrix  $K_{m_1}^{(1)}(X_1) K_{m_2}^{(2)}(X_2)$  the violation of inequality  $B \leq 2$  is sufficient condition to conclude that the two-mode state with the tomogram  $w(X_1, X_2, \theta_1, \theta_2)$  is entangled. It means that the optical tomogram of such entangled state can not be presented in the form of convex sum

$$w(X_1, X_2, \theta_1, \theta_2) = \sum_k p_k w_1^{(k)}(X_1, \theta_1) w_2^{(k)}(X_2, \theta_2) \quad (14)$$

where  $w_1^{(k)}(X_1, \theta_1)$  and  $w_2^{(k)}(X_2, \theta_2)$  are optical tomograms of the first and second mode states. These tomograms must satisfy also the Trifonov inequality (10).

## 5. Conclusions

To conclude we point out the main results of our work. We derived new inequalities for optical tomograms of quantum states which are obtained from Trifonov state-extended inequalities and presented in the form appropriate for experimental check by means of homodyne photon detection. We applied the recent qubit portrait method of studying qudit states to introduce a method to analyze entanglement of two-mode electromagnetic field state. Using the map of the optical tomogram of two-mode state onto analog of the spin-tomogram of two qubits we found that the violation of Bell inequalities for the obtained analog of spin tomogram is sufficient condition for the presence of the entanglement in the two-mode state under study.

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